Determine if each of the following converges or diverges.
If it converges, write "CONVERGES". If it diverges, write "DIVERGES".
Justify each answer using proper mathematical reasoning, algebra and/or calculus.

$$\sum_{k=1}^{\infty} \frac{3^{2k}(k!)^2}{(2k-1)!}$$

SCORE: _____ / 5 PTS

$$\lim_{k\to\infty} \frac{3^{2(k+1)}((k+1)!)^2}{(2(k+1)-1)!} \frac{(2k-1)!}{3^{2k}(k!)^2} = \lim_{k\to\infty} \frac{3^{2k+2}(k+1)^2(k!)^2}{(2k+1)!} \frac{(2k-1)!}{3^{2k}(k!)^2} = \lim_{k\to\infty} \frac{3^2(k+1)^2}{(2k+1)(2k)} = \frac{9}{4} > 1$$

So, $\sum_{k=1}^{\infty} \frac{3^{2k}(k!)^2}{(2k-1)!}$ diverges by Ratio Test

$$\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$$

SCORE: _____ / 3 PTS

$\frac{1}{2}$

$\frac{1}{2}$ $0 \le \frac{5^k}{4^k + 4^k} < \frac{5^k}{3^k + 4^k}$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

$\sum_{k=1}^{\infty} \frac{5^k}{4^k + 4^k} = \sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{5}{4}\right)^k$ diverges ($|r| = \frac{5}{4} > 1$) by Geometric Series Test, so $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$ diverges by Comparison Test

ALTERNATE SOLUTION – GRADE AGAINST ONLY ONE SOLUTION

$$\lim_{k\to\infty} \frac{5^k}{3^k + 4^k} = \lim_{k\to\infty} \frac{(\frac{5}{4})^k}{(\frac{3}{4})^k + 1} = \infty \text{ since } \lim_{k\to\infty}((\tfrac{3}{4})^k + 1) = 0 + 1 = 1 \text{ but } \lim_{k\to\infty}(\tfrac{5}{4})^k = \infty$$

So, $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$ diverges by Divergence Test

SEE ALTERNATE SOLUTION ON VERSION 1 KEY – GRADE AGAINST ONLY ONE SOLUTION

Determine if each of the following converges or diverges.
If it converges, write "CONVERGES". If it diverges, write "DIVERGES".
Justify each answer using proper mathematical reasoning, algebra and/or calculus.

$$\sum_{k=1}^{\infty} \frac{\sin 2k}{k+k^2}$$

SCORE: _____ / $6\frac{1}{2}$ **PTS**

$\frac{1}{2}$ $$0 < \left|\frac{\sin 2k}{k+k^2}\right| < \frac{1}{k+k^2} < \frac{1}{k^2}$$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges ($p = 2 > 1$) by p-Series Test, so $\sum_{k=1}^{\infty} \left|\frac{\sin 2k}{k+k^2}\right|$ converges by Comparison Test

and $\sum_{k=1}^{\infty} \frac{\sin 2k}{k+k^2}$ converges by Absolute Convergence Test

$$\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$$

SCORE: _____ / $9\frac{1}{2}$ **PTS**

$\frac{1}{2}$ $0 \le \frac{k \ln k}{(k+1)^3} < \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$ which is positive for $k \ge 2$, continuous for $k \ge 1$ $\frac{1}{2}$ $\frac{1}{2}$

and decreasing for $k \ge 2$ (since $\frac{d}{dx}\frac{\ln x}{x^2} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3} < 0$ for $x \ge \sqrt{e}$)

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{N\to\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big|_2^N = \lim_{N\to\infty}\left(-\frac{\ln N}{N} - \frac{1}{N} + \tfrac{1}{2}\ln 2 + \tfrac{1}{2}\right) = -0 - 0 + \tfrac{1}{2}\ln 2 + \tfrac{1}{2} = \tfrac{1}{2}\ln 2 + \tfrac{1}{2}$$ ie. converges

(since $\lim_{N\to\infty} \frac{\ln N}{N} = \lim_{N\to\infty} \frac{\frac{1}{N}}{1} = 0$)

So, $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ converges by Integral Test and $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$ converges by Comparison Test

$$\sum_{k=1}^{\infty} (1 - \tfrac{1}{k})^{k^2}$$

SCORE: _____ / 6 **PTS**

$$\lim_{k\to\infty} \sqrt[k]{(1-\tfrac{1}{k})^{k^2}} = \lim_{k\to\infty} (1-\tfrac{1}{k})^k = e^{-1} < 1$$

(since $\lim_{k\to\infty} \ln(1-\tfrac{1}{k})^k = \lim_{k\to\infty} k\ln(1-\tfrac{1}{k}) = \lim_{k\to\infty} \frac{\ln(1-\frac{1}{k})}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\frac{1}{1-\frac{1}{k}}(\frac{1}{k^2})}{-\frac{1}{k^2}} = \lim_{k\to\infty} -\frac{1}{1-\frac{1}{k}} = -\frac{1}{1-0} = -1$)

So, $\sum_{k=1}^{\infty} (1-\tfrac{1}{k})^{k^2}$ converges by Root Test